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LETTER TO THE EDITOR

Anomalous spectral statistics in a symmetrical billiard

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Abstract. It is a common assumption that time reversal invariance or some other anti-unitary symmetry for a classically chaotic system implies GOE-type spectral fluctuations for the corresponding quantum system. Based on previous work on structural invariance, we show that a time-reversal invariant system for certain point symmetries displays the GUE statistics typical of systems with broken time-reversal symmetry; specifically a billiard having only three-fold symmetry shows this unexpected behaviour.

A long-standing question concerns the connection between spectral statistics of quantum systems in the semiclassical limit on the one hand and the nature of their classical dynamics on the other. A large body of numerical evidence [1–4] has shown so far that classically chaotic systems with no further symmetries have spectra which are well described by Random Matrix Theory. This means that the spectral statistics are those of the corresponding random matrix ensembles. As is well known [5], there are three classical ensembles, depending on the behaviour of the system under time reversal. If the system is time-reversal invariant (TRI), the ensemble is one of real symmetric matrices known as the Gaussian orthogonal ensemble (GOE), whereas if it is not, the ensemble is one involving hermitean matrices known as the Gaussian unitary ensemble (GUE). The third ensemble (GSE) involves particles with spin and will not concern us in what follows. Further, numerical evidence has shown that the different invariant subspaces of a Hamiltonian with a symmetry (continuous or discrete) display the statistics individually and the corresponding results are statistically independent of one another.

Analytic discussions of such behaviour were first given, based on periodic orbit theory [6], and more recently [7, 8], based on the concepts of structural invariance and unitary representations of canonical transformations [9, 10].

Based on the second of these methods we came upon a particularity that contradicts the commonly accepted belief that systems with time-reversal invariance or some other anti-unitary symmetry [11] and a point symmetry have a spectrum with GOE statistics in each invariant subspace with respect to the point symmetry. The theory developed in [7, 8] will be applied to this situation and we shall find that, whenever the time-reversal operation and operations of the point group do not commute, the common assumption is wrong for non-self-conjugate invariant subspaces of the point group.

In [7, 8] the concept of a structural invariance group $\mathcal{H}$ is introduced as the group of canonical transformations that leaves all non-chaotic features of a canonical map C invariant; this implies the invariance of such things as invariant tori and discrete symmetries

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among others. We shall be particularly interested in the latter, and shall therefore consider a completely chaotic (K-) system that displays time-reversal invariance and some point symmetry. If $\mathcal{G}$ is the group of all bijective canonical transformations on the compact phase space we consider, then $\mathcal{H}$ is the subgroup of $\mathcal{G} \times \mathcal{G}$ that conserves these two symmetries and acts on C as:

$$(S, S') : C \to SCS'. \tag{1}$$

where S and $S' \in \mathcal{G}$ such that $(S, S') \in \mathcal{H}$. In the absence of any further constraint $\mathcal{H} = \mathcal{G} \times \mathcal{G}$. i.e. S and S' are arbitrary. The group $\mathcal{H}$ then generates from C a family Σ of canonical transformations. Using the representation theory of classical canonical transformations in quantum mechanics [10, 11] $\mathcal{G}$ is represented by $\mathcal{U}(n)$ where n is the number of cells of size h^d in our compact phase space of dimension $2d$. We shall use a suffix Q for the quantum counterparts of the classical quantities defined. The family Σ is represented by a family $\Sigma_Q = \mathcal{U}(n)$ which is uniquely endowed with a measure, namely the Haar measure. Thus the quantum counterpart of Σ has the pleasant property of forming an ensemble (for which we know that special and trivial elements such as e.g. unity are of measure zero). This ensemble is known as the CUE, which is the circular ensemble with the same local fluctuation properties as the GUE [12].

In general, the group $\mathcal{H}_Q$ has to be represented by a subgroup of $\mathcal{U}(n) \times \mathcal{U}(n)$, which will induce a unique measure on Σ_Q. For the specific case we are interested in, $\mathcal{H}$ is restricted both by time-reversal invariance and by some point symmetry.

The first implies the restriction $S' = TS^{-1}T$, where T is the time-reversal transformation. (This transformation is not canonical, but for any canonical transformation R, TRT is also canonical). The second implies $PS = SP$ and $PS' = S'P$ for all elements P of the point group $\mathcal{P}$. In the first case, the unitary representation yields Dyson's measure for the COE (the circular ensemble with local statistics equivalent to those of the GOE [12]). In the second case, $\mathcal{H}_Q$ is the direct product of $\mathcal{U}(n_f) \times \mathcal{U}(n_f)$ taken over all irreducible representations f of dimension $|f|$; n_f is determined such that the dimension of the invariant subspace transforming according to f is $n_f|f|$. This product structure of $\mathcal{H}$ leads to the independence of spectra pertaining to different invariant subspaces with a CUE for each.

The important question is how these two elements may combine. The common expectation is to find COE's with Dyson measures for each invariant subspace. This is certainly true if $TPT = P$ for all $P \in \mathcal{P}$. In the opposite case, however, things are somewhat more complicated. In the quantum version, the action of T on the invariant subspaces of $\mathcal{P}$ can be of two kinds: either T leaves them invariant or it permutes two subspaces among each other; the latter will happen for invariant subspaces corresponding to irreducible representations of P that are not self-conjugate. If the point group has only self-conjugate representations we are again reduced to the usual situation. We shall therefore focus on groups that have at least one conjugate pair of representations. Without loss of generality we can therefore limit our attention to a single conjugate pair of representations f, f^*. The quantum version of the structural invariance group for the invariant subspace corresponding to this pair of representations is given by $\mathcal{U}(n_f) \times \mathcal{U}(n_{f^*})$ with the action

$$(\Pi_f + \Pi_{f^*})C_Q \to U_1 \Pi_f C_Q U_2^t + U_2 \Pi_{f^*} C_Q U_1^t. \tag{2}$$

Here Π_f, Π_{f^*} are projectors onto the invariant subspaces corresponding to the irreducible representations f, f^* and U_1, U_2 are elements of $\mathcal{U}(n_f), \mathcal{U}(n_{f^*})$ respectively. Note that the transformations in the two unitary groups are independent and thus for one invariant subspace we have a Haar measure of $\mathcal{U}(n_f)$ leading to a CUE ensemble, that is, the ensemble that describes the subspaces does not reflect the TRI property of the whole system, but rather its violation in each subspace separately. Here we should mention the well known fact that such systems have a non-trivial degeneracy: since to every eigenvector of the subspace transforming according to f there corresponds a time-reversed vector of the one transforming according to f^*, and since the full system is invariant under time reversal, the eigenvalues of subspaces 1 and 2 must be degenerate. This property, which must certainly be true of the original system, is then easily seen to be conserved under the action given in equation (2), and holds for the entire ensemble.

It should be carefully noted that up to now we have only discussed the derivation of COE or CUE distribution for the *eigenphases* of periodically driven time-dependent systems. The generalization to bound systems is far from trivial and has been presented elsewhere [7]. We can, however, summarise the results obtained in these papers as follows: it is found that all properties that can be derived through such arguments as the above carry through for the eigenvalues of time-independent bound systems. The only caveat concerns the fact that the local eigenvalue density shows variations over large energy scales which are described by the Weyl formula, whereas eigenphases are always uniformly distributed on the unit circle. With this in mind, we can look for an example of the phenomenon described in the last paragraph, namely the appearance of invariant subspaces with GUE behaviour in TRI systems with appropriate discrete symmetry. The fact that such behaviour has never been reported before in numerical work is easily understood if one notes that parity, which is the discrete symmetry that appears most frequently, does indeed satisfy $TPT = P$. It therefore cannot give rise to invariant subspaces with GUE statistics. Since we are not aware of any other theoretical work making such a prediction, a numerical test is important.

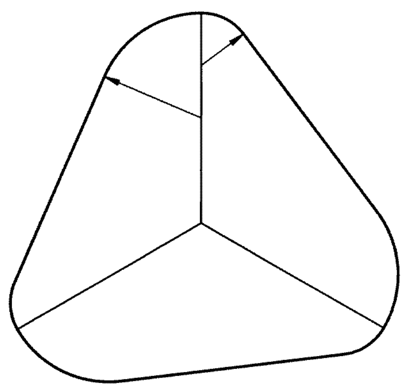
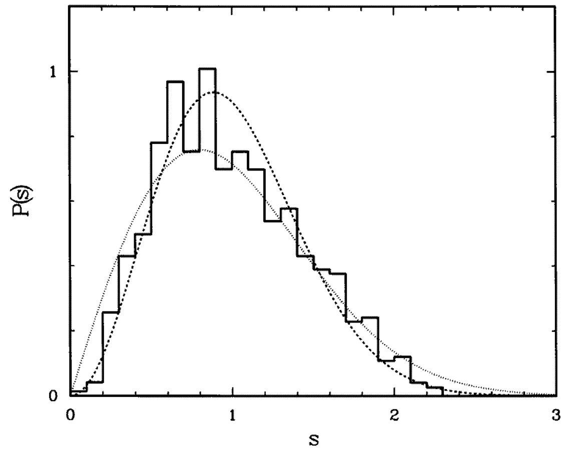
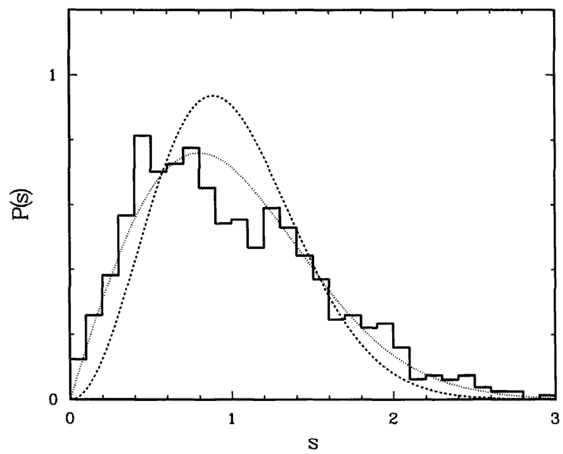


Figure 1. Form of the billiard studied in the text. The corners of the equilateral triangle are rounded by circle segments whose radii differ by a factor of two. A corresponding billiard with a mirror symmetry was also studied, in which the rounding was obtained by a circle segment corresponding to the larger radius.

To this end we consider the billiard shown in figure 1. It has three-fold symmetry, but no mirror symmetry. This last is to ensure that the symmetry group does have a pair of distinct but conjugate representations. In quantum mechanics, the problem separates into three invariant subspaces according to the value of the angular momentum taken mod 3, which is a good quantum number. Under complex conjugation, which corresponds to T, the subspaces corresponding to $m = 1$ and $m = -1$ mod 3 respectively are interchanged. In figure 2(a) we show the nearest-neighbour spacing distribution for a set of $\sim$ 800 eigenvalues of these subspaces (which were checked explicitly and found numerically to be



(*a*)



(*b*)

Figure 2. (*a*) The nearest-neighbour spacing distribution $P(S)$ for the billiard shown in figure 1. The agreement with the GUE prediction (thick dotted line) is apparent. (*b*) The nearest-neighbour spacing distribution $P(S)$ for the mirror-symmetric billiard. This shows ordinary GOE behaviour (thin dotted line). (*c*) The cumulative distribution $\int_0^S P(S')\mathrm{d}S'$ is shown, both for the billiard of figure 1 and for the mirror-symmetric billiard in order to show the short distance eigenvalue repulsion in greater detail. In all cases, the thick dotted line represents the GUE prediction, whereas the thin dots show the GOE prediction. The thick continuous line shows the data for the billiard of figure 1 and the thin line those of the mirror-symmetric billiard.

degenerate within the numerical accuracy). Both are clearly compatible with the GUE and strongly at variance with the GOE predictions. We also show in figure 2(*c*) the cumulative distribution for small distances. There it is quite clear that the type of level repulsion is indeed characteristic of the GUE for this billiard, just as we predict. In figure 3 we show the $\Delta_3(L)$ statistic which measures the long-range behaviour of the two-point correlation function of the spectrum. Again, agreement with the GUE prediction is good, at least for low values of L. A technical point: this particular billiard has a set of non-isolated marginally stable orbits, which affect the large-L behaviour of $\Delta_3(L)$. A standard technique

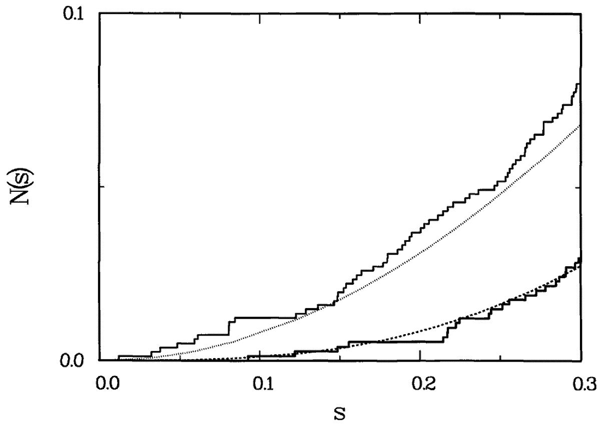


Figure 2. Continued.

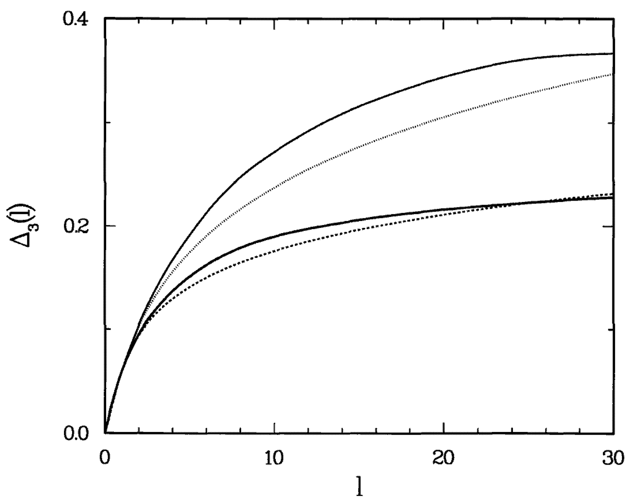


Figure 3. The $\Delta_3(L)$ statistic both for the billiard of figure 1 (thick continuous line) and the mirror-symmetric billiard (thin continuous line). Again the dotted lines indicate the GUE and GOE predictions respectively. The agreement at low L is satisfactory, whereas at large L discrepancies occur due to non-universal features of the spectrum such as long-range rigidity and non-isolated orbits. Nevertheless, it should be noted that the saturation value is approximately twice as large for the mirror-symmetric billiard as for the non-mirror-symmetric one, in good agreement with the difference expected between a GOE and a GUE.

(described in [13]) was used to eliminate this problem. Nevertheless, the agreement at large L remains slightly problematic, but the effect is sufficiently large and the small distance quadratic repulsion of the eigenvalues is seen with sufficient clarity to exclude the possibility of describing the data by a GOE.

Finally let us consider what happens in the case where the billiard does indeed have a mirror symmetry. In this case, all invariant subspaces of $\mathcal{P}$ are left invariant by T, so they are expected to show GOE statistics. As a check on the above, a mirror-symmetric form of the billiard shown in figure 1 was also studied. There it was found that the doubly degenerate subspace was rather well-described by a GOE. This is shown in figures 2(*b*) and 2(*c*), as well as in figure 3. As seen in those figures, both the short-range and the long-range behaviour of this billiard are markedly different from those of its non-mirror-symmetric analogue.

To summarise, we have applied a recent argument for the connection between chaos and spectral statistics based on the concept of structural invariance. We considered discrete symmetries in this context and obtained the unexpected result that time-reversal invariant systems may have invariant subspaces on which the spectrum has GUE fluctuations. This only occurs if the subspace invariant with respect to the point group is not invariant under time reversal. The prediction was confirmed numerically for a billiard with a threefold rotational symmetry. Apart from the obvious intrinsic interest of this finding, this successful prediction underscores the usefulness of the concept of structural invariance.

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